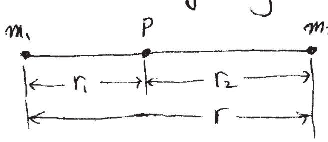
Center of Mass

The <u>center of mass</u> (c.m.) of a system of particles has many special properties, 4 of which are enumerated below:

1. The c.m. is the point where a rigid body can be supported in balance in a uniform gravitational field.
2. The c.m. of a system of particles will be at rest or move with constant velocity if & only if the only forces acting are internal (e.g. the 2 comps. of a binary star each rotate about their c.m.).
3. If there is an external force $\vec{F}$ on the system, we can write $\vec{F} = m\vec{a}_{cm}$, where m = the total mass of the system, and $\vec{a}_{cm}$ = the acc. of the c.m.
4. For a perfectly elastic collision between 2 particles viewed in the c.m. frame, each of the 2 particles simply reverses its velocity during the collision.

<u>Note</u>: property 2 follows from property 3, just as Newton's 1st law follows from the 2nd law

Consider 2 particles of mass m_1 & m_2 acting under their gravitational attraction, and no other force. One possible motion is that they will each rotate about a stationary pt. P on the line joining them. (P is the c.m.).



writing $F_{grav} = ma_c$ for each

$$\frac{m_1 m_2 G}{r^2} = m_1\left(\frac{4\pi^2 r_1}{T_1^2}\right)$$

these are = by Newton's 3rd law

$$\frac{m_1 m_2 G}{r^2} = m_2\left(\frac{4\pi^2 r_2}{T_2^2}\right)$$

Noting that $T_1 = T_2$, it is evident that

$$\boxed{m_1 r_1 = m_2 r_2} \quad \text{Eq. 1.}$$

<u>Note</u>: This derivation does not depend on the nature of the attractive force between m_1 & m_2. Also, if m_1 & m_2 were

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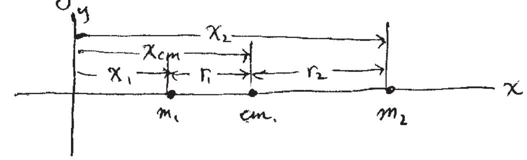
joined by a weightless rod, P would be the balance pt. (property 1).

In gen'l m_1 & m_2 are referred to a coordinate system:



$r_1 = x_{cm} - x_1$

$r_2 = x_2 - x_{cm}$

Substituting into Eq. 1: $\quad m_1(x_{cm} - x_1) = m_2(x_2 - x_{cm})$

$$\Rightarrow \boxed{x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}} \quad \text{Eq. 2.}$$

<u>Example</u>: Find x_{cm} of a 1-kg mass at (0,0) & a 3-kg mass at (4,0). (Ans. 3 m).

<u>Velocity & Acceleration of c.m.</u> If m_1 & m_2 are moving:

at time t: $\quad (m_1+m_2)\,x_{cm} = m_1 x_1 + m_2 x_2$

at time $t+\Delta t$: $\quad (m_1+m_2)\,x_{cm}' = m_1 x_1' + m_2 x_2'$

subtracting: $\quad (m_1+m_2)(x_{cm}' - x_{cm}) = m_1(x_1' - x_1) + m_2(x_2' - x_2)$

dividing by Δt: $\quad (m_1+m_2)\dfrac{(x_{cm}' - x_{cm})}{\Delta t} = m_1\dfrac{(x_1' - x)}{\Delta t} + m_2\dfrac{(x_2' - x_2)}{\Delta t}$

$$\Rightarrow \boxed{(m_1+m_2)\,v_{cm} = m_1 v_1 + m_2 v_2} \quad \text{Eq. 3.}$$

similarly,

$$(m_1+m_2)\,a_{cm} = m_1 a_1 + m_2 a_2 \quad \text{Eq. 4.}$$

<u>Note</u>: Another useful way of writing Eq. 3 is:

$$\boxed{v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}}$$

Eq. 3 has the important significance that $(m_1+m_2)v_{cm}$ is the total momentum $m_1v_1 + m_2v_2$ of the system. That is, a single particle of mass $m_1 + m_2$ = the total mass of the system, moving at velocity v_{cm} will have a momentum = to that of the whole system. Moreover, it follows

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from this that a system has zero total momentum in a frame of reference moving with velocity $= v_{cm}$, since the c.m. will appear stationary in this frame.

Newton's 2nd Law for a system of 2 particles: For 2 particles acted upon by internal and external forces:

$$F_1^{ext} + F_1^{int} = m_1 a_1$$
$$F_2^{ext} + F_2^{int} = m_2 a_2$$

adding:

$$\underbrace{F_1^{ext} + F_2^{ext}}_{F_{total}^{ext}} + \underbrace{F_1^{int} + F_2^{int}}_{=0 \text{ (by Newton's 3rd Law)}} = \underbrace{m_1 a_1 + m_2 a_2}_{(m_1+m_2)a_{cm} \text{ (from Eq. 4)}}$$

$$\Rightarrow \boxed{F_{tot}^{ext} = m_{tot}\, a_{cm}} \quad \text{Eq. 5.}$$

Note: Eq. 5 expresses properties 2 & 3. Property 4 will require a number of theorems to prove, as follows on the next 2 pages.

Example: Find the velocity of the c.m. for each of the following systems:

a) $m_1 = 2kg$ moving to the right at 1 m/s, & $m_2 = 1kg$ not moving.

b) $m_1 = 2kg$ moving to the right at 1 m/s, & $m_2 = 2kg$ moving to the left at 1 m/s.

c) $m_1 = 2kg$ moving to the right at 3 m/s, & $m_2 = 4kg$ moving to the right at 5 m/s.

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Conservation of Momentum & Energy Collisions in 1 Dimension

- Thm. 1. If momentum is conserved in one frame, it is also conserved in any other frame moving with const. v wrt the first.

Proof: before: $\underset{m_1}{\bullet}\xrightarrow{v_1}$ $\underset{m_2}{\bullet}\xrightarrow{v_2}$, after $\underset{m_1}{\bullet}\xrightarrow{v_1'}$ $\underset{m_2}{\bullet}\xrightarrow{v_2'}$

Cons. P: $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$.

In a frame moving at velocity v:

before: $\underset{m_1}{\bullet}\xrightarrow{v_1 - v}$ $\underset{m_2}{\bullet}\xrightarrow{v_2 - v}$, after: $\underset{m_1}{\bullet}\xrightarrow{v_1' - v}$ $\underset{m_2}{\bullet}\xrightarrow{v_2' - v}$

Does $m_1(v_1 - v) + m_2(v_2 - v) \overset{?}{=} m_1(v_1' - v) + m_2(v_2' - v)$

$$\Rightarrow m_1 v_1 - \cancel{m_1 v} + m_2 v_2 - \cancel{m_2 v} \overset{?}{=} m_1 v_1' - \cancel{m_1 v} + m_2 v_2' - \cancel{m_2 v}$$
$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \checkmark$$

- Thm. 2. The velocity of the c.m. is unchanged during a collision in which momentum is conserved.

Proof: before: $v_{cm} = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

after $v_{cm}' = \dfrac{m_1 v_1' + m_2 v_2'}{m_1 + m_2}$

If P is conserved then $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \Rightarrow v_{cm}' = v_{cm}$. ∎

- Corollary 1. From Thms 1 & 2 if momentum is conserved in one frame, it is also conserved in the c.m. frame.

- Thm. 3. In the c.m. frame the total momentum $P_{tot} = 0$.

Proof: In the c.m. frame we see: $\underset{m_1}{\xrightarrow{v_1 - v_{cm}}}$ $\underset{m_2}{\xrightarrow{v_2 - v_{cm}}}$

$$P_{tot} = m_1(v_1 - v_{cm}) + m_2(v_2 - v_{cm}) = m_1 v_1 + m_2 v_2 - (m_1 + m_2) v_{cm}$$
$$= m_1 v_1 + m_2 v_2 - (m_1 + m_2)\left[\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right] = 0 \quad \text{QED.}$$

Corollary 2. From corollary 1 & Thm. 3 we see that $P_{cm} = P_{cm}' = 0$. (Total momentum in c.m. frame is always zero.

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- **Thm. 4.** If momentum is conserved in a collision in one frame, then the loss of KE (if any) is the same in another frame moving at constant velocity v w.r.t. the first.

Proof: (Refer to "before"–"after" diagram in Thm. 1.)

in frame 1: $2\Delta KE_1 = m_1 v_1'^2 + m_2 v_2'^2 - m_1 v_1^2 - m_2 v_2^2$

in frame 2: $2\Delta KE_2 = m_2(v_1'-v)^2 + m_2(v_2'-v)^2 - m_1(v_1-v)^2 - m_2(v_2-v)^2$

$$= m_1 v_1'^2 + m_2 v_2'^2 - m_1 v_1^2 - m_2 v_2^2$$
$$+ m_1 v^2 + m_2 v_2^2 - m_1 v^2 - m_2 v^2$$
$$- 2v\left[m_1 v_1' + m_2 v_2' - m_1 v_1 - m_2 v_2\right]$$

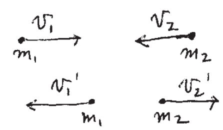
$\leftarrow [\ ] = 0$ because of cons. of P in frame 1.

$$= 2\Delta KE_1 \quad \text{Q.E.D.}$$

- **Corollary 3:** If momentum & KE are both conserved in one frame, they are also conserved in another frame moving at const. vel. w.r.t. the 1st frame, and specifically in the c.m. frame.

- **Thm 5.** The magnitude of each object's velocity is unchanged during an elastic collision as viewed in the c.m. frame.

Proof: in c.m. frame: before:



after:

Cons. KE: $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$

(from Corollary 1)
$$\begin{cases} P_{cm} = m_1 v_1 - m_2 v_2 = 0 \Rightarrow v_2 = \dfrac{m_1 v_1}{m_2} \\ P'_{cm} = m_1 v_1' - m_2 v_2' = 0 \Rightarrow v_2' = \dfrac{m_1 v_1'}{m_2} \end{cases}$$

Subst. $m_1 v_1^2 - m_2\left(\frac{m_1 v_1}{m_2}\right)^2 = m_1 v_1'^2 + m_2\left(\frac{m_1 v_1'}{m_2}\right)^2$

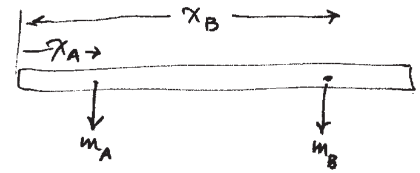
$$\left(m_1 + \frac{m_1^2}{m_2}\right) v_1^2 = \left(m_1 + \frac{m_1^2}{m_2}\right) v_1'^2$$

$$\left.\begin{aligned} v_1' &= \pm v_1 \\ v_2' &= \pm v_2 \end{aligned}\right\} \text{Q.E.D}$$

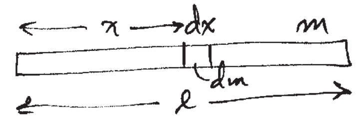
and similarly

* **Summary:** In any 1-dim. collision problem for which P & KE are both conserved, viewed in the c.m. frame each particle reverses its velocity during the collision.

Proof that the mass of an object can be thought of as concentrated at its c.m.



ruler has a mass m, ~~at~~ length l.



$dm = \frac{m}{l} dx$

Ruler alone: $x_{cm} = \dfrac{\int_0^l x\,dm}{m} = \dfrac{\int_0^l x \frac{m}{l} dx}{\cancel{m}} = \dfrac{1}{l} \dfrac{x^2}{2}\Big|_0^l$

$$= \frac{l}{2}.$$

ruler + masses m_A, m_B:

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_A x_A + m_B x_B + \int_0^l x\,dm}{m_A + m_B + m}$$

$$= \frac{m_A x_A + m_B x_B + m\left(\frac{l}{2}\right)}{m_A + m_B + m}$$

we are treating the effect of the ruler as a single particle m at the $l/2$ pt.